Qualitative structure of perturbations propagation process of the Fisher–Kolmogorov equation with a deviation of spatial variable

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Introduction

In 1937 Kolmogorov, Petrovskii and Piskunov [1] proposed the logistic equation with diffusion for simulate the propagation of genetically wave

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u[1-u], \tag{1}$$

In the same year Fisher [2] published the article devoted to the analysis of a similar equation.

1 *Kolmogorov A., Petrovsky I., Piscounov N.* Étude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique // Moscou Univ. Bull. Math., 1 (1937). P. 1–25.

2 *Fisher R. A.* The Wave of Advance of Advantageous Genes // Annals of Eugenics. 1937. V. 7. P. 355–369.

Introduction

Logistic equation generalization for simulation of population density distribution with dependencies of spatial and time deviations was considered in [1-3].

$$\frac{\partial u(t,x)}{\partial t} = \Delta u(t,x) + u(t,x)[1 + \alpha u(t,x) - (1 + \alpha (g * u)(t,x)] \qquad (2)$$

and convolution has following form

$$(g * u)(t,x) = \int\limits_{-\infty}^{t} \int\limits_{\Omega} g(t-\tau, x-y) u(\tau, y) dy d\tau, \qquad (3)$$

1. *Gourley S. A., So J. W.-H., Wu J. H.* Nonlocality of Reaction-Diffusion Equations Induced by Delay: Biological Modeling and Nonlinear Dynamics // Journal of Mathematical Sciences. 2004. Vol. 124, Issue 4. PP 5119–5153.
2. *Britton N. F.* Reaction-diffusion equations and their applications to biology / New York: Academic Press, 1986.
3. *Britton N. F.* Spatial structures and periodic travelling waves in an integro-differential reaction-diffusion population model // SIAM J. Appl. Math. 1990. V. 50. P. 1663–1688.

Logistic equation with a deviation of spatial variable

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u[1 - u(t, x - h)]. \tag{4}$$

$$u(t,x) = w(2t \pm x) \quad s = 2t \pm x$$

$$w'' - 2w' + w[1 - w(s - h)] = 0, \tag{5}$$

$$P(\lambda) \equiv \lambda^2 - 2\lambda - \exp(-h\lambda). \tag{6}$$

Logistic equation with a deviation of spatial variable

$$\begin{aligned} \lambda^2 - 2\lambda - \exp(-h\lambda) &= 0, \\ 2\lambda - 2 - h\exp(-h\lambda) &= 0. \end{aligned} \tag{7}$$

$$\lambda \approx -1.23141 \quad h = h_1^* \approx 1.12154$$

Lemma (1)

Quasipolynomial $P(\lambda)$ has one positive and two negative real roots at $0 < h < h_1^*$ and only one positive real root at $h > h_1^*$.

Lemma (2)

All roots of quasipolinom $P(\lambda)$ lie in the left half-plane for $0 < h < h_2^*$, except for one real positive root. Here $h_2^* = \frac{\arccos(-\sqrt{5}+2)}{\sqrt{\sqrt{5}-2}} \approx 3.72346$, The pair $\lambda = \pm i\omega_0$ of pure imaginary roots goes to the imaginary axis at $h = h_2^*$ and $\omega_0 = \sqrt{\sqrt{5} - 2} \approx 0.48587$.

Logistic equation with a deviation of spatial variable

$$h = h_2^* + \mu \quad 0 < \mu \ll 1$$

$$w(s,\mu) = 1 + \sqrt{\mu}\big(z(\tau)\exp(i\omega_0 s) + \bar{z}(\tau)\exp(-i\omega_0 s)\big) + \\ + \mu w_1(s,\tau) + \mu^{3/2} w_2(s,\tau) + \ldots, \quad \tau = \mu s, w_j(s,\tau)(j=1,2) \quad (8)$$

$$\frac{dz}{d\tau} = \varphi_0 z + \varphi_1 |z|^2 z, \quad (9)$$

at $\varphi_0 = \dfrac{2\omega_0^2(-1 + i\omega_0)}{P'(i\omega_0)}$,

$$\varphi_1 = \frac{1}{P'(i\omega_0)}\left(2\omega_0^2(1 - \omega_0^2 - 2i\omega_0) + \beta\left((\omega_0^2 + 2i\omega_0)^2 - \frac{1}{\omega_0^2 + 2i\omega_0}\right)\right),$$

$$\beta = \frac{\omega_0^2 + 2i\omega_0}{4\omega_0^2 + 4i\omega_0 + (\omega_0^2 + 2i\omega_0)^2}.$$

$$\varphi_0 \approx 0.136807 - 0.20660i \quad \varphi_1 \approx -0.04429 - 0.03664i$$

Lemma (3)

Let $h = h_2^* + \mu$ *and* $0 < \mu \ll 1$ *then there exists* $\mu_0 > 0$ *such that for all* $0 < \mu < \mu_0$ *equation* (5) *has dichotomous cycle which one-dimensional unstable manifold and following asymptotic*

$$\sqrt{-\mathrm{Re}\,(\varphi_0)/\mathrm{Re}\,(\varphi_1)}\exp\Big(i\varepsilon s\big(\mathrm{Im}\,(\varphi_0)\mathrm{Re}\,(\varphi_1)-\mathrm{Re}\,(\varphi_0)\mathrm{Im}\,(\varphi_1)\big)/\mathrm{Re}\,(\varphi_0)+i\gamma\Big)$$

and γ *— is an arbitrary constant, which determines the phase shift along the cycle.*

$$u(t,x) = u(t,x+T), \quad T > 0 \tag{10}$$

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} - v(t,x-h), \quad v(t,x) = v(t,x+T). \tag{11}$$

$$v(t,x) = \exp\lambda \exp i\omega x$$

$$\lambda = -\omega^2 - \exp i\omega h. \tag{12}$$

$$h^* = 2.791544, \quad \omega^* = 0.88077. \tag{13}$$

$$T = 2\pi/\omega^* \quad h = h^* + \varepsilon$$

$$u(t,x,\varepsilon) = 1 + \sqrt{\varepsilon}u_0(t,\tau,x) + \varepsilon u_1(t,\tau,x) + \varepsilon^{3/2}u_2(t,\tau,x) + \ldots, \tag{14}$$

and $\tau = \varepsilon t$,

$$u_0(t,\tau,x) = z(\tau)\exp\big(i(\omega_0 t + \omega^* x)\big) + \bar{z}(\tau)\exp\big(-i(\omega_0 t + \omega^* x)\big), \quad \omega_0 = \sin\omega^* h^*.$$

$$\frac{dz}{d\tau} = \varphi_0 z + \varphi_1 |z|^2 z, \tag{15}$$

$$\varphi_0 = i\omega^* \exp(-i\omega^* h^*),$$
$$\varphi_1 = 2\cos\omega^* h^* \big(1 + \exp(-i\omega^* h^*)\big) - \big(\exp(-2i\omega^* h^*) + \exp(i\omega^* h^*)\big) w_2.$$
$$\varphi_0 \approx 0.5558 - 0.6833i, \quad \varphi_1 \approx -0.1701 + 0.59i.$$

Lemma (4)

Let $h = h^* + \varepsilon$ *then there exists* $\varepsilon_0 > 0$ *such that for all* $0 < \varepsilon < \varepsilon_0$ *boundary value problem* (4), (10) *has orbitally asymptotically stable cycle with following asymptotic*

$$\sqrt{-\mathrm{Re}\,(\varphi_0)/\mathrm{Re}\,(\varphi_1)}\exp\Big(i\varepsilon t\big(\mathrm{Im}\,(\varphi_0)\mathrm{Re}\,(\varphi_1) - \mathrm{Re}\,(\varphi_0)\mathrm{Im}\,(\varphi_1)\big)/\mathrm{Re}\,(\varphi_0) + i\gamma\Big)$$

and γ *— is an arbitrary constant, which determines the phase shift along the cycle.*

Numerical analysis

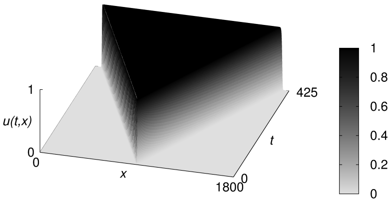
$$\dot{u}_j = \frac{u_{j+1} - 2u_j + u_{j-1}}{(\Delta x)^2} + \big[1 - u_{j-k}\big]u_j, \tag{16}$$

$j = 0, \ldots, N-1,\ k = \lfloor h/\Delta x \rfloor$

$$N = 1.8 \cdot 10^5 \quad N = 1.8 \cdot 10^6$$

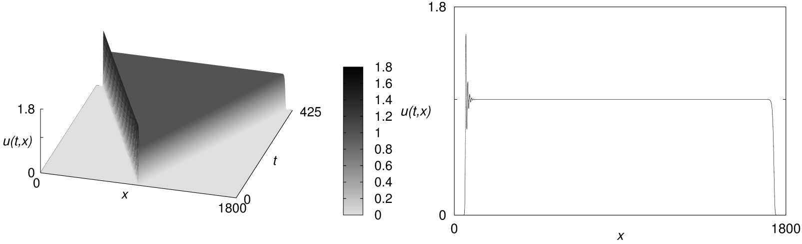
$$u_j(0) = \begin{cases} 0.1, \ \text{if } j \in [89950, 90050], \\ 0, \ \text{otherwise.} \end{cases} \tag{17}$$

$h = 1.2$



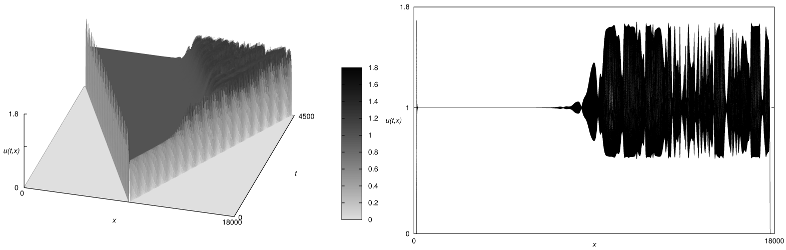
Wave propagation in logistic equation with spatial variable deviation $h = 1.2$ and cross-section $t = 425$

$h = 2.7$



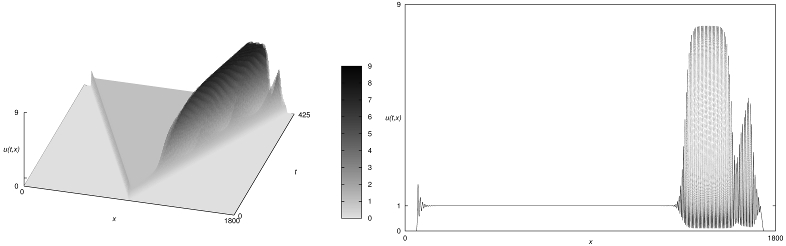
Wave propagation in logistic equation with spatial variable deviation $h = 2.7$ and cross-section $t = 425$

$h = 2.81$



Wave propagation in logistic equation with spatial variable deviation $h = 2.81$ and cross-section $t = 4500$

movie



Wave propagation in logistic equation with spatial variable deviation $h = 3$ and cross-section $t = 425$

Thank you for attention!